CHAPTER 14

QUADRATIC EQUATIONS

14.1 INTRODUCTION

In earlier classes, we have studied about quadratic equations with real coefficients and real roots only. In this chapter, we shall study about quadratic equations with real coefficients and complex roots. We shall also discuss quadratic equations with complex coefficients and their solutions in the complex number system. But, let us first recall some definitions and results.

14.2 SOME USEFUL DEFINITIONS AND RESULTS

REAL POLYNOMIAL *Let $a_0, a_1, a_2, \ldots, a_n$ be real numbers and x is a real variable. Then, $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$ is called a real polynomial of real variable x with real coefficients.*

For example, $x^2 - 4x + 3$, $2x^3 - 6x^2 + 11x - 5$ etc. are real polynomials.

COMPLEX POLYNOMIAL *If $a_0, a_1, a_2, \ldots, a_n$ are complex numbers and x is a varying complex number, then $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$ is called a complex polynomial or a polynomial of complex variable with complex coefficients.*

For example, $2x^2 - (3 + 7i)x + (9i - 3)$, $x^3 - 5i\, x^2 + (1 - 2i)x + (3 + 4i)$ etc are complex polynomials.

DEGREE OF A POLYNOMIAL *A polynomial $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$, real or complex, is a polynomial of degree n, if $a_n \neq 0$.*

The polynomials $2x^3 - 7x^2 + x + 5$, $(3 - 2i)x^2 - i\,x + 5$ are polynomials of degree 3 and 2 respectively.

A polynomial of second degree is generally called a quadratic polynomial and polynomials of degree 3 and 4 are known as *cubic* and *biquadratic* polynomials.

POLYNOMIAL EQUATION *If $f(x)$ is a polynomial, then $f(x) = 0$ is called a polynomial equation.*

If $f(x)$ is a quadratic polynomial, then $f(x) = 0$ is called a quadratic equation. The general form of a *quadratic equation* is $ax^2 + bx + c = 0,\ a \neq 0$.

Here, x is the variable and a, b, c are called coefficients real or imaginary.

ROOTS OF AN EQUATION *The values of the variable satisfying a given equation are called its roots.*

Thus, $x = \alpha$, is a root of the equation $f(x) = 0$, if $f(\alpha) = 0$.

For example, $x = 1$ is a root of the equation $x^3 - 6x^2 + 11x - 6 = 0$, because

$$1^3 - 6 \times 1^2 + 11 \times 1 - 6 = 1 - 6 + 11 - 6 = 0$$

Similarly, $x = \omega$ and $x = \omega^2$ are roots of the equation $x^2 + x + 1 = 0$ as they satisfy it.

SOLUTION SET *The set of all roots of an equation, in a given domain, is called the solution set of the equation.*

For example, the set $\{1, 2, 3\}$ is the solution set of the equation $x^3 - 6x^2 + 11x - 6 = 0$.

Solving an equation means finding its solution set. In other words, solving an equation is the process of obtaining its all roots.

IDENTITY *An expression involving equality and a variable is called an identity, if it is satisfied by every value of the variable.*

For example, $x^2 - 9 = (x-3)(x+3)$ is an identity as it is satisfied by every value of x.

and, $\frac{(x-a)(x-b)}{(c-a)(c-b)} + \frac{(x-b)(x-c)}{(a-b)(a-c)} + \frac{(x-c)(x-a)}{(b-c)(b-a)} = 1$ is also an identity as it holds good for all values of x.

FUNDAMENTAL THEOREM OF ALGEBRA *Every polynomial equation $f(x) = 0$ has at least one root, real or imaginary (complex).*

Thus, $x^7 - 3x^5 + 2x^2 = x + 2 = 0$ has at least one root. But, $f(x) = \sqrt{x} + 3 = 0$ has no root as this equation is not a polynomial equation. Fundamental theorem does not apply on this equation.

The fundamental theorem guarantees for one root of a polynomial equation. The following theorem states about the exact number of roots of a polynomial equation.

THEOREM *Every polynomial equation $f(x) = 0$ of degree n has exactly n roots real or imaginary.*

14.3 QUADRATIC EQUATION

The general form of a quadratic equation is $ax^2 + bx + c = 0$, $a \neq 0$ where a, b, c are numbers (real or complex) and x is a variable.

The following theorem suggests about the number of roots of a quadratic equation.

THEOREM *A quadratic equation cannot have more than two roots.*

PROOF If possible, let α, β, γ be three distinct roots of the quadratic equation $ax^2 + bx + c = 0$, where $a, b, c \in R$ and $a \neq 0$. Then, each one of α, β, γ will satisfy this equation.

$$\therefore \quad a\alpha^2 + b\alpha + c = 0 \qquad \text{...(i)}$$

$$a\beta^2 + b\beta + c = 0 \qquad \text{...(ii)}$$

$$\text{and,} \quad a\gamma^2 + b\gamma + c = 0 \qquad \text{...(iii)}$$

Subtracting (ii) from (i), we obtain

$$a(\alpha^2 - \beta^2) + b(\alpha - \beta) = 0$$

$$\Rightarrow \quad (\alpha - \beta)\{a(\alpha + \beta) + b\} = 0$$

$$\Rightarrow \quad a(\alpha + \beta) + b = 0 \qquad [\because \alpha \text{ and } \beta \text{ are distinct } \therefore \alpha - \beta \neq 0] \qquad \text{...(iv)}$$

Subtracting (iii) from (ii), we obtain

$$a(\beta^2 - \gamma^2) + b(\beta - \gamma) = 0$$

$$\Rightarrow \quad (\beta - \gamma)\{a(\beta + \gamma) + b\} = 0$$

$$\Rightarrow \quad a(\beta + \gamma) + b = 0 \qquad [\because \beta \text{ and } \gamma \text{ are distinct } \therefore \beta - \gamma \neq 0] \qquad \text{...(v)}$$

Subtracting (v) from (iv), we get : $a(\alpha - \gamma) = 0$. But, this is not possible, because α and γ are distinct and $a \neq 0$. So, their product cannot be zero.

Thus, the assumption that a quadratic equation has three distinct real roots is wrong.

Hence, a quadratic equation cannot have more than 2 roots.

Q.E.D.

REMARK *It follows from the above theorem that if a quadratic equation is satisfied by more than two values of x, then it is satisfied by every value of x and so it is an identity.*

14.4 QUADRATIC EQUATIONS WITH REAL COEFFICIENTS

In earlier classes, we have solved quadratic equations with real coefficients and real roots either by factorization or by using Sridharacharya's formula. In this section, we shall mainly concentrate on quadratic equations with real coefficients and complex roots.

Consider the quadratic equation

$$ax^2 + bx + c = 0 \qquad \text{...(i)}$$

where $a, b, c \in R$ and $a \neq 0$.

Multiplying both sides of (i) by a, we get

$$a^2x^2 + abx + ac = 0$$

$$\Rightarrow \quad a^2x^2 + abx + \frac{b^2}{4} = \frac{b^2}{4} - ac$$

$$\Rightarrow \quad \left(ax + \frac{b}{2}\right)^2 = \frac{b^2 - 4ac}{4}$$

$$\Rightarrow \quad ax + \frac{b}{2} = \pm \frac{\sqrt{b^2 - 4ac}}{2}$$

$$\Rightarrow \quad ax = -\frac{b}{2} \pm \frac{\sqrt{b^2 - 4ac}}{2} \Rightarrow ax = \frac{-b \pm \sqrt{b^2 - 4ac}}{2} \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Thus, the quadratic equation $ax^2 + bx + c = 0$, where $a, b, c \in R$ and $a \neq 0$ has two roots, say α and β, given by

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Now, if we look at these roots, we observe that the roots depend upon the value of the quantity $b^2 - 4ac$. This quantity is generally denoted by D and is known as the *discriminate* of the quadratic equation (i). We also observe the following results:

RESULT I *If $b^2 - 4ac = 0$ i.e. $D = 0$, then $\alpha = \beta = -\frac{b}{2a}$.*

Thus, if $b^2 - 4ac = 0$, then the quadratic equation has real and equal roots each equal to $-b/2a$.

RESULT II *If a, b, c are rational numbers and $b^2 - 4ac$ is positive and a perfect square, then $\sqrt{b^2 - 4ac}$ is a rational number and hence α and β are rational and unequal.*

Thus, if $a, b, c \in Q$ and $b^2 - 4ac$ is positive and a perfect square, then roots are rational and unequal. If $a, b, c \in R$ and $b^2 - 4ac$ is positive and a perfect square, then roots are real and distinct.

RESULT III *If $b^2 - 4ac > 0$ i.e. $D > 0$ but it is not a perfect square, then roots are irrational and unequal.*

REMARK *If $a, b, c \in Q$ and $b^2 - 4ac$ is positive but not a perfect square, then roots are irrational and they always occur in conjugate pair like $2 + \sqrt{3}$ and $2 - \sqrt{3}$. However, if a, b, c are irrational numbers and $b^2 - 4ac$ is positive but not a perfect square, then the roots may not occur in conjugate pairs. For example, the roots of the equation $x^2 - (5 + \sqrt{2})x + 5\sqrt{2} = 0$ are 5 and $\sqrt{2}$ which do not form a conjugate pair.*

RESULT IV *If $b^2 - 4ac < 0$ i.e. $D < 0$, then $4ac - b^2 > 0$ and so the roots are imaginary and are given by*

$$\alpha = \frac{-b + i\sqrt{4ac - b^2}}{2a} \text{ and } \beta = \frac{-b - i\sqrt{4ac - b^2}}{2a}$$

Clearly, α and β are complex conjugate of each other i.e. $\alpha = \bar{\beta}$ and $\bar{\alpha} = \beta$.

REMARK *If $b^2 - 4ac < 0$, then the roots are complex conjugate of each other. In fact, complex roots of an equation with real coefficients always occur in conjugate pairs like $2 + 3i$ and $2 - 3i$. However, this may not be true in case of equations with complex coefficients. For example, $x^2 - 2ix - 1 = 0$ has both roots equal to i.*

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Solve the equation $4x^2 + 9 = 0$ by factorization method.*

SOLUTION We have,

$$4x^2 + 9 = 0$$
$$\Rightarrow \quad 4x^2 - 9i^2 = 0$$
$$\Rightarrow \quad (2x)^2 - (3i)^2 = 0$$
$$\Rightarrow \quad (2x + 3i)(2x - 3i) = 0$$
$$\Rightarrow \quad 2x + 3i = 0 \text{ or, } 2x - 3i = 0$$
$$\Rightarrow \quad x = -\frac{3}{2}i, \text{ or, } x = \frac{3}{2}i$$

Hence, the roots of the given equation are $\frac{3}{2}i$ and $-\frac{3}{2}i$.

EXAMPLE 2 *Solve the equation $x^2 - 4x + 13 = 0$ by factorization method.*

SOLUTION We have,

$$x^2 - 4x + 13 = 0$$
$$\Rightarrow \quad x^2 - 4x + 4 + 9 = 0$$
$$\Rightarrow \quad (x-2)^2 + 9 = 0$$
$$\Rightarrow \quad (x-2)^2 - 9i^2 = 0$$
$$\Rightarrow \quad (x-2)^2 - (3i)^2 = 0$$
$$\Rightarrow \quad \{(x-2) - 3i\}\{(x-2) + 3i\} = 0$$
$$\Rightarrow \quad (x - 2 - 3i)(x - 2 + 3i) = 0$$
$$\Rightarrow \quad x - 2 - 3i = 0, \text{ or } x - 2 + 3i = 0$$
$$\Rightarrow \quad x = 2 + 3i, \text{ or } x = 2 - 3i$$

Hence, the roots of the given equation are $2 + 3i$ and $2 - 3i$.

EXAMPLE 3 *Solve the equation $9x^2 - 12x + 20 = 0$ by factorization method only.*

SOLUTION We have,

$$9x^2 - 12x + 20 = 0$$
$$\Rightarrow \quad 9x^2 - 12x + 4 + 16 = 0$$
$$\Rightarrow \quad (3x-2)^2 + 16 = 0$$
$$\Rightarrow \quad (3x-2)^2 - 16i^2 = 0$$
$$\Rightarrow \quad \{(3x-2) + 4i\}\{(3x-2) - 4i\} = 0$$
$$\Rightarrow \quad (3x - 2 + 4i)(3x - 2 - 4i) = 0$$
$$\Rightarrow \quad 3x - 2 + 4i = 0, \text{ or } 3x - 2 - 4i = 0$$
$$\Rightarrow \quad 3x = 2 - 4i, \text{ or } 3x = 2 + 4i$$

$$\Rightarrow \quad x = \frac{2}{3} - \frac{4}{3}i \text{ or } x = \frac{2}{3} + \frac{4}{3}i$$

Hence, the roots of the given equation are $\frac{2}{3} - \frac{4}{3}i$ and $\frac{2}{3} + \frac{4}{3}i$.

EXAMPLE 4 *Solve the quadratic equation $2x^2 - 4x + 3 = 0$ by using the general expressions for the roots of a quadratic equation.*

SOLUTION Comparing the given equation with the general form $ax^2 + bx + c = 0$, we get

$$a = 2, b = -4 \text{ and } c = 3$$

Substituting the values of a, b, c in $\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $\beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$, we get

$$\alpha = \frac{4 + \sqrt{16 - 24}}{4} \quad \text{and,} \quad \beta = \frac{4 - \sqrt{16 - 24}}{4}$$
$$\Rightarrow \quad \alpha = \frac{4 + \sqrt{-8}}{4} \quad \text{and,} \quad \beta = \frac{4 - \sqrt{-8}}{4}$$
$$\Rightarrow \quad \alpha = \frac{4 + 2\sqrt{2}i}{4} \quad \text{and,} \quad \beta = \frac{4 - 2\sqrt{2}i}{4}$$
$$\Rightarrow \quad \alpha = 1 + \frac{1}{\sqrt{2}}i \quad \text{and,} \quad \beta = 1 - \frac{1}{\sqrt{2}}i$$

Hence, the roots of the given equation are $1 + \frac{1}{\sqrt{2}}i$ and $1 - \frac{1}{\sqrt{2}}i$.

EXAMPLE 5 *Solve the equation $25x^2 - 30x + 11 = 0$ by using the general expression for the roots of a quadratic equation.*

SOLUTION Comparing the given equation with the general form of the quadratic equation $ax^2 + bx + c = 0$, we get: $a = 25$, $b = -30$ and $c = 11$.

Substituting these values in $\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $\beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$, we get

$$\alpha = \frac{30 + \sqrt{900 - 1100}}{50} \quad \text{and} \quad \beta = \frac{30 - \sqrt{900 - 1100}}{50}$$
$$\Rightarrow \quad \alpha = \frac{30 + \sqrt{-200}}{50} \quad \text{and} \quad \beta = \frac{30 - \sqrt{-200}}{50}$$
$$\Rightarrow \quad \alpha = \frac{30 + 10i\sqrt{2}}{50} \quad \text{and} \quad \beta = \frac{30 - 10i\sqrt{2}}{50}$$
$$\Rightarrow \quad \alpha = \frac{3}{5} + \frac{\sqrt{2}}{5}i \quad \text{and} \quad \beta = \frac{3}{5} - \frac{\sqrt{2}}{5}i$$

Hence, the roots of the given equation are $\frac{3}{5} \pm \frac{\sqrt{2}}{5}i$.

EXERCISE 14.1

LEVEL-1

Solve the following quadratic equations by factorization method only (1-5):

1. $x^2 + 1 = 0$
2. $9x^2 + 4 = 0$
3. $x^2 + 2x + 5 = 0$
4. $4x^2 - 12x + 25 = 0$ [NCERT]
5. $x^2 + x + 1 = 0$

Solve the following quadratics (6-18):

6. $4x^2 + 1 = 0$
7. $x^2 - 4x + 7 = 0$
8. $x^2 + 2x + 2 = 0$
9. $5x^2 - 6x + 2 = 0$
10. $21x^2 + 9x + 1 = 0$
11. $x^2 - x + 1 = 0$
12. $x^2 + x + 1 = 0$ [NCERT]
13. $17x^2 - 8x + 1 = 0$
14. $27x^2 - 10x + 1 = 0$ [NCERT]
15. $17x^2 + 28x + 12 = 0$
16. $21x^2 - 28x + 10 = 0$ [NCERT]
17. $8x^2 - 9x + 3 = 0$
18. $13x^2 + 7x + 1 = 0$
19. $2x^2 + x + 1 = 0$ [NCERT]
20. $\sqrt{3}x^2 - \sqrt{2}\,x + 3\sqrt{3} = 0$ [NCERT]
21. $\sqrt{2}\,x^2 + x + \sqrt{2} = 0$ [NCERT]
22. $x^2 + x + \dfrac{1}{\sqrt{2}} = 0$ [NCERT]
23. $x^2 + \dfrac{x}{\sqrt{2}} + 1 = 0$ [NCERT]
24. $\sqrt{5}x^2 + x + \sqrt{5} = 0$ [NCERT]
25. $-x^2 + x - 2 = 0$ [NCERT]
26. $x^2 - 2x + \dfrac{3}{2} = 0$ [NCERT]
27. $3x^2 - 4x + \dfrac{20}{3} = 0$ [NCERT]

ANSWERS

1. $i, -i$
2. $\dfrac{2}{3}i, \dfrac{-2}{3}i$
3. $-1 + 2i, -1 - 2i$
4. $\dfrac{3}{2} + 2i, \dfrac{3}{2} - 2i$
5. $-\dfrac{1}{2} + i\dfrac{\sqrt{3}}{2}, -\dfrac{1}{2} - i\dfrac{\sqrt{3}}{2}$
6. $\dfrac{1}{2}i, \dfrac{-1}{2}i$
7. $2 \pm \sqrt{3}i$
8. $-1 \pm i$
9. $\dfrac{3}{5} \pm \dfrac{1}{5}i$
10. $\dfrac{-3}{14} \pm \dfrac{i\sqrt{3}}{42}$
11. $\dfrac{1}{2} \pm \dfrac{i\sqrt{3}}{2}$
12. $-\dfrac{1}{2} \pm \dfrac{i\sqrt{3}}{2}$
13. $\dfrac{4}{17} \pm \dfrac{1}{17}i$
14. $\dfrac{5}{27} \pm \dfrac{\sqrt{2}}{27}i$
15. $\dfrac{-14}{17} \pm \dfrac{2\sqrt{2}}{17}i$
16. $\dfrac{2}{3} \pm \dfrac{\sqrt{14}}{21}i$
17. $\dfrac{9}{16} \pm \dfrac{\sqrt{15}}{16}i$
18. $\dfrac{-7}{26} \pm \dfrac{\sqrt{3}}{26}i$
19. $\dfrac{-1 \pm \sqrt{7}i}{4}$
20. $\dfrac{\sqrt{2} \pm \sqrt{34}i}{2\sqrt{3}}$
21. $\dfrac{-1 \pm \sqrt{7}i}{2\sqrt{2}}$
22. $\dfrac{-1 \pm \sqrt{2\sqrt{2} - 1}\; i}{2}$
23. $\dfrac{-1 \pm \sqrt{7}i}{2\sqrt{2}}$
24. $\dfrac{-1 \pm \sqrt{19}i}{2\sqrt{5}}$
25. $\dfrac{-1 \pm \sqrt{7}i}{-2}$
26. $1 \pm \dfrac{1}{\sqrt{2}}i$
27. $\dfrac{2}{3} \pm \dfrac{4}{3}i$

HINTS TO NCERT & SELECTED PROBLEMS

5. We have,

$$x^2 + x + 1 = 0$$

$$\Rightarrow \quad x^2 + 2(x)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + 1 = 0$$

$$\Rightarrow \quad \left(x + \frac{1}{2}\right)^2 + \frac{3}{4} = 0$$

$$\Rightarrow \quad \left(x + \frac{1}{2}\right)^2 - \frac{3}{4}i^2 = 0$$

$$\Rightarrow \quad \left(x + \frac{1}{2}\right)^2 - \left(\frac{\sqrt{3}}{2}i\right)^2 = 0$$

$$\Rightarrow \quad \left(x + \frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\left(x + \frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = 0$$

$$\Rightarrow \quad \left(x + \frac{1 + \sqrt{3}i}{2}\right)\left(x + \frac{1 - \sqrt{3}\,i}{2}\right) = 0$$

$$\Rightarrow \quad x + \frac{1 + \sqrt{3}i}{2} = 0 \text{ or, } x + \frac{1 - \sqrt{3}i}{2} = 0$$

$$\Rightarrow \quad x = -\frac{1 + i\sqrt{3}}{2} \text{ or, } x = \frac{-1 + i\sqrt{3}}{2}$$

12. We have, $x^2 + x + 1 = 0$

Comparing the given equation with the general form $ax^2 + bx + c = 0$, we get

$a = 1, b = 1$ and $c = 1$

Substituting the values of a, b, c in

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}, \text{ we get}$$

$$\alpha = \frac{-1 + \sqrt{1 - 4}}{2} \text{ and } \beta = \frac{-1 - \sqrt{1 - 4}}{2}$$

$$\Rightarrow \quad \alpha = \frac{-1 + i\sqrt{3}}{2} \text{ and } \beta = \frac{-1 - i\sqrt{3}}{2}$$

14. We have, $27x^2 - 10x + 1 = 0$

Comparing this equation with $ax^2 + bx + c = 0$, we get: $a = 27, b = -10, c = 1$

Substituting the values of a, b, c in

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}, \text{ we get}$$

$$\Rightarrow \quad \alpha = \frac{10 + \sqrt{100 - 108}}{54} \text{ and } \beta = \frac{10 - \sqrt{100 - 108}}{54}$$

$$\Rightarrow \quad \alpha = \frac{10 + \sqrt{-8}}{54} \text{ and } \beta = \frac{10 - \sqrt{-8}}{54}$$

$$\Rightarrow \quad \alpha = \frac{5 + i\sqrt{2}}{27} \text{ and } \beta = \frac{5 - i\sqrt{2}}{27}$$

16. The given equation is $21x^2 - 28x + 10 = 0$. Comparing this equation with $ax^2 + bx + c = 0$, we get: $a = 21, b = -28, c = 10$

Substituting these values in

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}, \text{ we get}$$

$$\alpha = \frac{28+\sqrt{784-840}}{42} \text{ and } \beta = \frac{28-\sqrt{784-840}}{42}$$

$$\Rightarrow \alpha = \frac{28+\sqrt{-56}}{42} \text{ and } \beta = \frac{28-\sqrt{-56}}{42}$$

$$\Rightarrow \alpha = \frac{2}{3}+\frac{i\sqrt{14}}{21} \text{ and } \beta = \frac{2}{3}-\frac{i\sqrt{14}}{21}$$

20. The given equation is $\sqrt{3}x^2-\sqrt{2}x+3\sqrt{3}=0$. Comparing this equation with $ax^2+bx+c=0$, we get: $a=\sqrt{3}, b=-\sqrt{2}, c=3\sqrt{3}$.

Substituting these values in

$$\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } \beta = \frac{-b-\sqrt{b^2-4ac}}{2a}, \text{ we get}$$

$$\alpha = \frac{\sqrt{2}+\sqrt{2-36}}{2\sqrt{3}} \text{ and } \beta = \frac{\sqrt{2}-\sqrt{2-36}}{2\sqrt{3}}$$

$$\Rightarrow \alpha = \frac{\sqrt{2}+i\sqrt{34}}{2\sqrt{3}} \text{ and } \beta = \frac{\sqrt{2}-i\sqrt{34}}{2\sqrt{3}}$$

21. The given equation is $\sqrt{2}x^2+x+\sqrt{2}=0$. Comparing this equation with $ax^2+bx+c=0$, we get: $a=\sqrt{2}, b=1, c=\sqrt{2}$.

Substituting these values in

$$\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } \beta = \frac{-b-\sqrt{b^2-4ac}}{2a}, \text{ we get}$$

$$\alpha = \frac{-1+\sqrt{1-8}}{2\sqrt{2}} \text{ and } \beta = \frac{-1-\sqrt{1-8}}{2\sqrt{2}}$$

$$\Rightarrow \alpha = \frac{-1+i\sqrt{7}}{2\sqrt{2}} \text{ and } \beta = \frac{-1-i\sqrt{7}}{2\sqrt{2}}$$

22. The given equation is $x^2+x+\frac{1}{\sqrt{2}}=0$. Comparing this equation with $ax^2+bx+c=0$, we get: $a=1, b=1, c=\frac{1}{\sqrt{2}}$.

Substituting these values in

$$\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } \beta = \frac{-b-\sqrt{b^2-4ac}}{2a}, \text{ we get}$$

$$\Rightarrow \alpha = \frac{-1+\sqrt{1-2\sqrt{2}}}{2} \text{ and } \beta = \frac{-1-\sqrt{1-2\sqrt{2}}}{2}$$

$$\Rightarrow \alpha = \frac{-1+i\sqrt{2\sqrt{2}-1}}{2} \text{ and } \beta = \frac{-1-i\sqrt{2\sqrt{2}-1}}{2}$$

23. The given equation is $x^2+\frac{x}{\sqrt{2}}+1=0$. Comparing this equation with $ax^2+bx+c=0$, we get: $a=1, b=\frac{1}{\sqrt{2}}$ and $c=1$.

Substituting these values in

$$\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } \beta = \frac{-b-\sqrt{b^2-4ac}}{2a}, \text{ we get}$$

$$\alpha = \frac{-\frac{1}{\sqrt{2}}+\sqrt{\frac{1}{2}-4}}{2} \text{ and } \beta = \frac{-\frac{1}{\sqrt{2}}-\sqrt{\frac{1}{2}-4}}{2}$$

$$\Rightarrow \alpha = \frac{-1+i\sqrt{7}}{2\sqrt{2}} \text{ and } \beta = \frac{-1-i\sqrt{7}}{2\sqrt{2}}$$

24. The given equation is $\sqrt{5}x^2+x+\sqrt{5}=0$. Comparing this equation with ax^2+bx+c, we get $a=\sqrt{5}, b=1$ and $c=\sqrt{5}$. Substituting these values in

$$\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } \beta = \frac{-b-\sqrt{b^2-4ac}}{2a}, \text{ we get}$$

$$\alpha = \frac{-1+\sqrt{1-20}}{2\sqrt{5}} \text{ and } \beta = \frac{-1-\sqrt{1-20}}{2\sqrt{5}}$$

$$\Rightarrow \alpha = \frac{-1+i\sqrt{19}}{2\sqrt{5}} \text{ and } \beta = \frac{-1-i\sqrt{19}}{2\sqrt{5}}$$

25. The given equation is $-x^2+x-2=0$. Comparing this equation with ax^2+bx+c, we get: $a=-1, b=1$ and $c=-2$. Substituting these values in

$$\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } \beta = \frac{-b-\sqrt{b^2-4ac}}{2a}, \text{ we get}$$

$$\Rightarrow \alpha = \frac{-1+\sqrt{1-8}}{-2} \text{ and } \beta = \frac{-1-\sqrt{1-8}}{-2}$$

$$\Rightarrow \alpha = \frac{-1+i\sqrt{7}}{-2} \text{ and } \beta = \frac{-1-i\sqrt{7}}{-2}$$

$$\Rightarrow \alpha = \frac{1}{2}-i\frac{\sqrt{7}}{2} \text{ and } \beta = \frac{1}{2}+i\frac{\sqrt{7}}{2}$$

26. The given equation is $x^2-2x+\frac{3}{2}=0$. Comparing this equation with $ax^2+bx+c=0$, we get: $a=1, b=-2$ and $c=3/2$.

Substituting these values in

$$\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } \beta = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

$$\Rightarrow \alpha = \frac{2+\sqrt{4-6}}{2} \text{ and } \beta = \frac{2-\sqrt{4-6}}{2}$$

$$\Rightarrow \alpha = 1+\frac{i}{\sqrt{2}} \text{ and } \beta = 1-\frac{i}{\sqrt{2}}$$

27. The given equation is $3x^2-4x+\frac{20}{3}=0$. Com,paring this equation with $ax^2+bx+c=0$, we get: $a=3, b=-4$ and $c=\frac{20}{3}$. Substituting these values in

$$\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } \beta = \frac{-b-\sqrt{b^2-4ac}}{2a}, \text{ we get}$$

$$\alpha = \frac{4+\sqrt{16-80}}{6} \text{ and } \beta = \frac{4-\sqrt{16-80}}{6}$$

$$\Rightarrow \quad \alpha = \frac{4+8i}{6} \text{ and } \beta = \frac{4-8i}{6}$$

$$\Rightarrow \quad \alpha = \frac{2}{3} + \frac{4}{3}i \text{ and } \beta = \frac{2}{3} - \frac{4}{3}i$$

14.5 QUADRATIC EQUATIONS WITH COMPLEX COEFFICIENTS

Consider the quadratic equation

$$ax^2 + bx + c = 0 \qquad \text{...(i)}$$

where a, b, c are complex numbers and $a \neq 0$.

Proceeding as in section 14.4, we obtain that the roots of equation (i) are given by

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and,} \quad \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

These roots are complex as a, b, c are complex numbers.

Since the order relation is not defined in case of complex numbers. Therefore, we cannot assign positive or negative sign to the discriminate $D = b^2 - 4ac$. However, equation (i) has complex roots which are equal, if $D = b^2 - 4ac = 0$ and unequal roots if $D = b^2 - 4ac \neq 0$.

REMARK *In case of quadratic equations with real coefficients imaginary (complex) roots always occur in conjugate pairs. However, it is not true for quadratic equations with complex coefficients. For example, the equation $4x^2 - 4ix - 1 = 0$ has both roots equal to $\frac{1}{2}i$.*

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Solve the following quadratic equations by factorization method:*

(i) $x^2 - 5ix - 6 = 0$ (ii) $x^2 + 4ix - 4 = 0$

SOLUTION (i) The given equation is

$$\begin{aligned}
&& x^2 - 5ix - 6 &= 0 \\
\Rightarrow && x^2 - 5ix + 6i^2 &= 0 \\
\Rightarrow && x^2 - 3ix - 2ix + 6i^2 &= 0 \\
\Rightarrow && x(x-3i) - 2i(x-3i) &= 0 \\
\Rightarrow && (x-3i)(x-2i) &= 0 \\
\Rightarrow && x - 3i = 0, x - 2i &= 0 \\
\Rightarrow && x = 3i, x &= 2i
\end{aligned}$$

Hence, the roots of the given equation are $3i$ and $2i$.

(ii) We have,

$$\begin{aligned}
&& x^2 + 4ix - 4 &= 0 \\
\Rightarrow && x^2 + 4ix + 4i^2 &= 0 \\
\Rightarrow && (x+2i)^2 &= 0 \\
\Rightarrow && x + 2i &= 0 \quad \text{(twice)} \\
\Rightarrow && x &= -2i, -2i
\end{aligned}$$

Hence, both the roots of the equation are equal to $-2i$.

EXAMPLE 2 *Solve the following equations by factorization method*

(i) $x^2 - \sqrt{2}i\,x + 12 = 0$ (ii) $3x^2 + 7ix + 6 = 0$ (iii) $x^2 - (3\sqrt{2} + 2i)\,x + 6\sqrt{2}i = 0$

SOLUTION (i) We have,

$$\begin{aligned}
&& x^2 - \sqrt{2}\,ix + 12 &= 0 \\
\Rightarrow && x^2 - 3\sqrt{2}i\,x + 2\sqrt{2}\,i\,x - 12i^2 &= 0 \\
\Rightarrow && x(x - 3\sqrt{2}i) + 2\sqrt{2}i(x - 3\sqrt{2}i) &= 0 \\
\Rightarrow && (x - 3\sqrt{2}\,i)(x + 2\sqrt{2}\,i) &= 0 \\
\Rightarrow && x - 3\sqrt{2}i = 0 \text{ or, } x + 2\sqrt{2}\,i &= 0 \\
\Rightarrow && x = 3\sqrt{2}\,i \text{ or, } x &= -2\sqrt{2}\,i
\end{aligned}$$

Hence, the roots of the given equation are $-2\sqrt{2}\,i$ and $3\sqrt{2}\,i$.

(ii)

$$\begin{aligned}
&& 3x^2 + 7ix + 6 &= 0 \\
\Rightarrow && 3x^2 + 9ix - 2ix - 6i^2 &= 0 \\
\Rightarrow && 3x(x+3i) - 2i(x+3i) &= 0 \\
\Rightarrow && (x+3i)(3x-2i) &= 0 \\
\Rightarrow && x + 3i = 0 \text{ or, } 3x - 2i &= 0 \\
\Rightarrow && x = -3i \text{ or, } x &= \frac{2}{3}i
\end{aligned}$$

Hence, the roots of the given equation are $-3i$ and $\frac{2}{3}i$.

(iii)

$$\begin{aligned}
&& x^2 - (3\sqrt{2} + 2i)\,x + 6\sqrt{2}\,i &= 0 \\
\Rightarrow && (x^2 - 3\sqrt{2}x) - (2ix - 6\sqrt{2}i) &= 0 \\
\Rightarrow && x(x - 3\sqrt{2}) - 2i(x - 3\sqrt{2}) &= 0 \\
\Rightarrow && (x-2i)(x-3\sqrt{2}) &= 0 \\
\Rightarrow && x - 2i = 0, x - 3\sqrt{2} &= 0 \\
\Rightarrow && x &= 2i \text{ or, } 3\sqrt{2}
\end{aligned}$$

Hence, the roots of the given equation are $2i$ and $3\sqrt{2}$.

EXAMPLE 3 *Solve the following quadratic equations by using the general expressions for the roots of a quadratic equation:*

(i) $x^2 - (3\sqrt{2} - 2i)\,x - 6\sqrt{2}\,i = 0$ (ii) $2x^2 + 3i\,x + 2 = 0$

SOLUTION (i) On comparing the given equation with the general equation $ax^2 + bx + c = 0$, we get: $a = 1, b = -(3\sqrt{2} - 2i)$ and $c = -6\sqrt{2}i$.

Substituting the values of a, b, c in $\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $\beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$, we get

$$\alpha = \frac{(3\sqrt{2} - 2i) + \sqrt{(3\sqrt{2} - 2i)^2 + 24\sqrt{2}i}}{2}, \text{ and } \beta = \frac{(3\sqrt{2} - 2i) - \sqrt{(3\sqrt{2} - 2i)^2 + 24\sqrt{2}i}}{2}$$

$$\Rightarrow \quad \alpha = \frac{(3\sqrt{2} - 2i) + \sqrt{(3\sqrt{2} + 2i)^2}}{2}, \quad \text{and} \quad \beta = \frac{(3\sqrt{2} - 2i) - \sqrt{13\sqrt{2} + 2i)^2}}{2}$$

$$\Rightarrow \quad \alpha = \frac{3\sqrt{2} - 2i + 3\sqrt{2} + 2i}{2}, \quad \text{and} \quad \beta = \frac{(3\sqrt{2} - 2i) - (3\sqrt{2} + 2i)}{2}$$

$$\Rightarrow \quad \alpha = 3\sqrt{2}, \beta = -2i$$

Hence, the roots of the given equation are $3\sqrt{2}$ and $-2i$.

(ii) On comparing the given equation with the general equation $ax^2+bx+c=0$, we get $a=2, b=3i$ and $c=2$.

Substituting these values of a, b, c in $\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a}$ and $\beta = \frac{-b-\sqrt{b^2-4ac}}{2a}$, we get

$$\alpha = \frac{-3i+\sqrt{9i^2-16}}{4} \quad \text{and} \quad \beta = \frac{-3i-\sqrt{9i^2-16}}{4}$$

$$\Rightarrow \quad \alpha = \frac{-3i+\sqrt{-25}}{4} \quad \text{and} \quad \beta = \frac{-3i-\sqrt{-25}}{4}$$

$$\Rightarrow \quad \alpha = \frac{-3i+5i}{4} \quad \text{and} \quad \beta = \frac{-3i-5i}{4}$$

$$\Rightarrow \quad \alpha = \frac{i}{2} \quad \text{and} \quad \beta = -2i$$

Hence, the roots of the given equation are $\frac{i}{2}$ and $-2i$.

LEVEL-2

EXAMPLE 4 *Solve:* $2x^2-(3+7i)\,x-(3-9i) = 0.$

SOLUTION On comparing the given equation with the general form $ax^2+bx+c=0$, we obtain $a=2, b=-(3+7i), c=-(3-9i)$.

Substituting the values of a, b, c in $\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a}$ and $\beta = \frac{-b-\sqrt{b^2-4ac}}{2a}$, we get

$$\alpha = \frac{(3+7i)+\sqrt{(3+7i)^2+8\,(3-9i)}}{4} \quad \text{and,} \quad \beta = \frac{(3+7i)-\sqrt{(3+7i)^2+8\,(3-9i)}}{4}$$

$$\Rightarrow \alpha = \frac{(3+7i)+\sqrt{9-49+42i+24-72i}}{4} \quad \text{and,} \quad \beta = \frac{(3+7i)-\sqrt{9-49+42i+24-72i}}{4}$$

$$\Rightarrow \alpha = \frac{3+7i+\sqrt{-16-30i}}{4} \quad \text{and,} \quad \beta = \frac{3+7i-\sqrt{-16-30i}}{4} \qquad \text{...(i)}$$

Let us now find $\sqrt{-16-30\,i}$.

Let $a+ib = \sqrt{-16-30i}$. Then,

$$a+ib = \sqrt{-16-30i}$$

$$\Rightarrow \quad a^2-b^2+2i\,ab = -16-30i$$

$$\Rightarrow \quad a^2-b^2 = -16 \qquad \text{...(ii)}$$

$$\text{and,} \quad 2\,ab = -30 \qquad \text{...(iii)}$$

$$\therefore \quad (a^2+b^2)^2 = (a^2-b^2)^2+4\,a^2\,b^2$$

$$\Rightarrow \quad (a^2+b^2)^2 = 256+900 = 1156$$

$$\Rightarrow \quad a^2+b^2 = 34$$

$$\text{Now,} \quad a^2-b^2 = -16 \text{ and } a^2+b^2 = 34$$

$$\Rightarrow \quad a^2 = 9 \text{ and } b^2 = 25$$

$$\Rightarrow \quad a = \pm 3 \text{ and } b = \pm 5$$

From (iii), we find that a and b are of opposite signs.

$\therefore \quad a = 3$ and $b = -5$ or, $a = -3$ and $b = 5$.

Hence, $\sqrt{-16-30i} = 3-5i$ or, $-3+5i$.

Substituting either of these values in (i), we get

$$\alpha = \frac{(3+7i)+(3-5i)}{4} \quad \text{and,} \quad \beta = \frac{(3+7i)-(3-5i)}{4}$$

$$\Rightarrow \quad \alpha = \frac{3}{2}+\frac{1}{2}i \quad \text{and,} \quad \beta = 3i$$

Hence, the roots of the given equation are $\frac{3}{2}+\frac{1}{2}i$ and $3i$.

EXAMPLE 5 *Solve:* $x^2-(7-i)\,x+(18-i) = 0$ *over C.*

SOLUTION Comparing the given equation with the general form $ax^2+bx+c=0$, we get $a=1, b=-(7-i)$ and $c=18-i$. Substituting these values in

$$\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and, } \beta = \frac{-b-\sqrt{b^2-4ac}}{2a}, \text{ we get}$$

$$\alpha = \frac{(7-i)+\sqrt{(7-i)^2-4\,(18-i)}}{2}, \quad \beta = \frac{(7-i)-\sqrt{(7-i)^2-4\,(18-i)}}{2}$$

$$\Rightarrow \quad \alpha = \frac{(7-i)+\sqrt{-24-10i}}{2}, \quad \beta = \frac{(7-i)-\sqrt{-24-10i}}{2} \qquad \text{...(i)}$$

Let us now find $\sqrt{-24-10i}$.

Let, $a+ib = \sqrt{-24-10i}$. Then,

$$(a+ib)^2 = -24-10i$$

$$\Rightarrow \quad (a^2-b^2)+2i\,ab = -24-10i$$

$$\Rightarrow \quad a^2-b^2 = -24 \qquad \text{...(ii)}$$

$$\text{and,} \quad 2ab = -10 \qquad \text{...(iii)}$$

$$\text{Now,} \quad (a^2+b^2)^2 = (a^2-b^2)^2+4\,a^2\,b^2$$

$$\Rightarrow \quad (a^2+b^2)^2 = 576+100 = 676 \Rightarrow a^2+b^2 = 26 \qquad \text{...(iv)}$$

Solving (ii) and (iii), we get $a=\pm 1$ and $b=\pm 5$.

From (iii), we find that ab is negative.

$\therefore \quad a = 1, b = -5$ or, $a = -1, b = 5$.

$\therefore \quad a+ib = 1-5i$ or, $-1+5i$

Hence, $\sqrt{-24-10i} = \pm(1-5i)$

Substituting either of these values in (i), we get

$$\alpha = \frac{7-i+1-5i}{2} = 4-3i \text{ and } \beta = \frac{(7-i)-(1-5i)}{2} = 3+2i$$

Hence, the roots of the given equation are $4-3i$ and $3+2i$.

EXERCISE 14.2

1. Solving the following quadratic equations by factorization method:
   (i) $x^2+10ix-21 = 0$ (ii) $x^2+(1-2i)\,x-2i = 0$
   (iii) $x^2-(2\sqrt{3}+3i)\,x+6\sqrt{3}i = 0$ (iv) $6x^2-17ix-12 = 0$
2. Solve the following quadratic equations:
   (i) $x^2-(3\sqrt{2}+2i)\,x+6\sqrt{2}\,i = 0$ (ii) $x^2-(5-i)\,x+(18+i)=0$
   (iii) $(2+i)\,x^2-(5-i)\,x+2\,(1-i)=0$ (iv) $x^2-(2+i)\,x-(1-7i)=0$
   (v) $i\,x^2-4x-4i = 0$ (vi) $x^2+4ix-4 = 0$

(vii) $2x^2 + \sqrt{15}\, ix - i = 0$ [NCERT] (viii) $x^2 - x + (1 + i) = 0$

(ix) $ix^2 - x + 12i = 0$ [NCERT] (x) $x^2 - (3\sqrt{2} - 2i)\, x - \sqrt{2}\, i = 0$

(xi) $x^2 - (\sqrt{2} + i)\, x + \sqrt{2}i = 0$ [NCERT] (xii) $2x^2 - (3 + 7i)\, x + (9i - 3) = 0$

ANSWERS

1. (i) $-3i, -7i$ (ii) $-1, 2i$ (iii) $2\sqrt{3}, 3i$ (iv) $\frac{3}{2}i, \frac{4}{3}i$

2. (i) $3\sqrt{2}, 2i$ (ii) $3 - 4i, 2 + 3i$ (iii) $1 - i, \frac{4}{5} - \frac{2}{5}i$ (iv) $3 - i, -1 + 2i$

(v) $-2i, -2i$ (vi) $-2i, -2i$ (vii) $\frac{1 + (4 - \sqrt{15})\, i}{4}, \frac{-1 - (\sqrt{15} + 4)\, i}{4}$

(viii) $1 - i, i$ (ix) $-4i, 3i$ (x) $\frac{3\sqrt{2} - 2i}{2} \pm \frac{4 - \sqrt{2}i}{2}$

(xi) $\sqrt{2}, i$ (xii) $\frac{3 + i}{2}, 3i$

HINTS TO NCERT & SELECTED PROBLEMS

2. (vii) The given equation is $2x^2 + \sqrt{15}\, i\, x - i = 0$.

Comparing this equation with the standard equation $ax^2 + bx + c = 0$, we get

$a = 2, b = \sqrt{15}i$ and $c = -i$

Substituting these values in

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and, } \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}, \text{ we get}$$

$$\alpha = \frac{-\sqrt{15}\, i + \sqrt{-15 + 8i}}{4} \text{ and, } \beta = \frac{-\sqrt{15}\, i - \sqrt{-15 + 8\, i}}{4}$$

Let $\sqrt{-15 + 8i} = a + ib$. Then,

$-15 + 8i = (a + ib)^2$

$\Rightarrow -15 + 8i = a^2 - b^2 + 2i\, ab$

$\Rightarrow a^2 - b^2 = -15$ and $2ab = 8$

Now, $(a^2 + b^2)^2 = (a^2 - b^2)^2 + 4a^2 b^2$

$\Rightarrow (a^2 + b^2)^2 = (-15)^2 + 64 = 289$

$\Rightarrow a^2 + b^2 = 17$

Solving $a^2 - b^2 = -15$ and $a^2 + b^2 = 17$, we get

$a^2 = 1$ and $b^2 = 16$

$\Rightarrow a = \pm 1$ and $b = \pm 4$

$\Rightarrow a = 1, b = 4$ or $a = -1, b = -4$ [$\because ab = 4 > 0 \therefore a$ and b are of the same sign]

$\therefore \sqrt{-15 + 8i} = 1 + 4i, -1 - 4i$

When $\sqrt{-15 + 8i} = 1 + 4i$:

$$\alpha = \frac{\sqrt{15}i + 1 + 4i}{4} = \frac{1 + (\sqrt{15} + 4)\, i}{4} \text{ and } \beta = \frac{-\sqrt{15}i - (1 + 4i)}{2} = -\frac{1 - (\sqrt{5} + 4)\, i}{2}$$

When $\sqrt{-15 + 8i} = -1 - 4i$:

$$\alpha = \frac{\sqrt{15}\, i - 1 - 4i}{2} = \frac{-1 - (\sqrt{15} + 4)\, i}{2} \text{ and } \beta = \frac{-\sqrt{15}i + 1 + 4i}{2} = \frac{1 + (4 - \sqrt{15})\, i}{2}$$

(ix) The given equation is $i\, x^2 - x + 12i = 0$.

Comparing this equation with the standard equation $ax^2 + bx + c = 0$, we get $a = i, b = -1$ and $c = 12i$.

Substituting these values in

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}, \text{ we get}$$

$$\alpha = \frac{1 + \sqrt{1 + 48}}{2i} \text{ and } \beta = \frac{1 - \sqrt{1 + 48}}{2i}$$

$$\Rightarrow \alpha = \frac{1 + 7}{2i} \text{ and } \beta = \frac{1 - 7}{2i}$$

$$\Rightarrow \alpha = \frac{4}{i} \text{ and } \beta = -\frac{3}{i} \Rightarrow \alpha = 0 - 4i \text{ and } \beta = 3i$$

(xi) The given equation is $x^2 - (\sqrt{2} + i)\, x + \sqrt{2}\, i = 0$.

Comparing this equation with the standard equation $ax^2 + bx + c = 0$, we get $a = 1, b = -(\sqrt{2} + i)$ and $c = \sqrt{2}\, i$.

Substituting these values in

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}, \text{ we get}$$

$$\alpha = \frac{(\sqrt{2} + i) + \sqrt{(\sqrt{2} + i)^2 - 4\sqrt{2}\, i}}{2} \text{ and } \beta = \frac{(\sqrt{2} + i) - \sqrt{(\sqrt{2} + i)^2 - 4\sqrt{2}\, i}}{2}$$

$$\Rightarrow \alpha = \frac{\sqrt{2} + i + \sqrt{(\sqrt{2} - i)^2}}{2} \text{ and } \beta = \frac{(\sqrt{2} + i) - \sqrt{(\sqrt{2} - i)^2}}{2}$$

$$\Rightarrow \alpha = \frac{\sqrt{2} + i + \sqrt{2} - i}{2} \text{ and } \beta = \frac{(\sqrt{2} + i) - (\sqrt{2} - i)}{2}$$

$$\Rightarrow \alpha = \sqrt{2} \text{ and } \beta = i$$

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. Write the number of real roots of the equation $(x - 1)^2 + (x - 2)^2 + (x - 3)^2 = 0$.
2. If a and b are roots of the equation $x^2 - px + q = 0$, then write the value of $\frac{1}{a} + \frac{1}{b}$.
3. If roots α, β of the equation $x^2 - px + 16 = 0$ satisfy the relation $\alpha^2 + \beta^2 = 9$, then write the value of p.
4. If $2 + \sqrt{3}$ is a root of the equation $x^2 + px + q = 0$, then write the values of p and q.

5. If the difference between the roots of the equation $x^2 + ax + 8 = 0$ is 2, write the values of a.
6. Write the roots of the equation $(a-b)\,x^2 + (b-c)\,x + (c-a) = 0$.
7. If a and b are roots of the equation $x^2 - x + 1 = 0$, then write the value of $a^2 + b^2$.
8. Write the number of quadratic equations, with real roots, which do not change by squaring their roots.
9. If α, β are roots of the equation $x^2 + lx + m = 0$, write an equation whose roots are $-\frac{1}{\alpha}$ and $-\frac{1}{\beta}$.
10. If α, β are roots of the equation $x^2 - a(x+1) - c = 0$, then write the value of $(1+\alpha)(1+\beta)$.

ANSWERS

1. No real root 2. $\frac{1}{q}$ 3. ± 8 4. $p = -4, q = 1$ 5. ± 8 6. $1, \frac{c-a}{a-b}$
7. -1 8. 3 9. $mx^2 - lx + 1 = 0$ 10. $1 - c$

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. The complete set of values of k, for which the quadratic equation $x^2 - kx + k + 2 = 0$ has equal roots, consists of
   (a) $2 + \sqrt{12}$ (b) $2 \pm \sqrt{12}$ (c) $2 - \sqrt{12}$ (d) $-2 - \sqrt{2}$
2. For the equation $|x|^2 + |x| - 6 = 0$, the sum of the real roots is
   (a) 1 (b) 0 (c) 2 (d) none of these
3. If a, b are the roots of the equation $x^2 + x + 1 = 0$, then $a^2 + b^2 =$
   (a) 1 (b) 2 (c) -1 (d) 3
4. If α, β are roots of the equation $4x^2 + 3x + 7 = 0$, then $1/\alpha + 1/\beta$ is equal to
   (a) $7/3$ (b) $-7/3$ (c) $3/7$ (d) $-3/7$
5. The values of x satisfying $\log_3 (x^2 + 4x + 12) = 2$ are
   (a) $2, -4$ (b) $1, -3$ (c) $-1, 3$ (d) $-1, -3$
6. The number of real roots of the equation $(x^2 + 2x)^2 - (x+1)^2 - 55 = 0$ is
   (a) 2 (b) 1 (c) 4 (d) none of these
7. If α, β are the roots of the equation $ax^2 + bx + c = 0$, then $\frac{1}{a\alpha + b} + \frac{1}{a\beta + b} =$
   (a) c/ab (b) a/bc (c) b/ac (d) none of these
8. If α, β are the roots of the equation $x^2 + px + 1 = 0$; γ, δ the roots of the equation $x^2 + qx + 1 = 0$, then $(\alpha - \gamma)(\alpha + \delta)(\beta - \gamma)(\beta + \delta) =$
   (a) $q^2 - p^2$ (b) $p^2 - q^2$ (c) $p^2 + q^2$ (d) none of these
9. The number of real solutions of $|2x - x^2 - 3| = 1$ is
   (a) 0 (b) 2 (c) 3 (d) 4

10. The number of solutions of $x^2 + |x - 1| = 1$ is
    (a) 0 (b) 1 (c) 2 (d) 3
11. If x is real and $k = \frac{x^2 - x + 1}{x^2 + x + 1}$, then
    (a) $k \in [1/3, 3]$ (b) $k \geq 3$ (c) $k \leq 1/3$ (d) none of these
12. If the roots of $x^2 - bx + c = 0$ are two consecutive integers, then $b^2 - 4c$ is
    (a) 0 (b) 1 (c) 2 (d) none of these
13. The value of a such that $x^2 - 11x + a = 0$ and $x^2 - 14x + 2a = 0$ may have a common root is
    (a) 0 (b) 12 (c) 24 (d) 32.
14. The values of k for which the quadratic equation $kx^2 + 1 = kx + 3x - 11x^2$ has real and equal roots are
    (a) $-11, -3$, (b) $5, 7$ (c) $5, -7$ (d) none of these
15. If the equations $x^2 + 2x + 3\lambda = 0$ and $2x^2 + 3x + 5\lambda = 0$ have a non-zero common roots, then $\lambda =$
    (a) 1 (b) -1 (c) 3 (d) none of these
16. If one root of the equation $x^2 + px + 12 = 0$ is 4, while the equation $x^2 + px + q = 0$ has equal roots, the value of q is
    (a) $49/4$ (b) $4/49$ (c) 4 (d) none of these
17. The value of p and q $(p \neq 0, q \neq 0)$ for which p, q are the roots of the equation $x^2 + px + q\ ab = 0$ are
    (a) $p = 1, q = -2$ (b) $p = -1, q = -2$
    (c) $p = -1, q = 2$ (d) $p = 1, q = 2$
18. The set of all values of m for which both the roots of the equation $x^2 - (m+1)x + m + 4 = 0$ are real and negative, is
    (a) $(-\infty, -3] \cup [5, \infty)$ (b) $[-3, 5]$
    (c) $(-4, -3]$ (d) $(-3, -1]$
19. The number of roots of the equation $\frac{(x+2)(x-5)}{(x-3)(x+6)} = \frac{x-2}{x+4}$ is
    (a) 0 (b) 1 (c) 2 (d) 3
20. If α and β are the roots of $4x^2 + 3x + 7 = 0$, then the value of $\frac{1}{\alpha} + \frac{1}{\beta}$ is
    (a) $\frac{4}{7}$ (b) $-\frac{3}{7}$ (c) $\frac{3}{7}$ (d) $-\frac{3}{4}$
21. If, α, β are the roots of the equation $x^2 + px + q = 0$, then $-\frac{1}{\alpha}, -\frac{1}{\beta}$ are the roots of the equation
    (a) $x^2 - px + q = 0$ (b) $x^2 + px + q = 0$
    (c) $qx^2 + px + 1 = 0$ (d) $qx^2 - px + 1 = 0$
22. If the difference of the roots of $x^2 - px + q = 0$ is unity, then
    (a) $p^2 + 4q = 1$ (b) $p^2 - 4q = 1$
    (c) $p^2 + 4q^2 = (1 + 2q)^2$ (d) $4p^2 + q^2 = (1 + 2p)^2$

23. If α, β are the roots of the equation $x^2 - p(x+1) - c = 0$, then $(\alpha+1)(\beta+1) =$

(a) c (b) $c-1$ (c) $1-c$ (d) none of these

24. The least value of k which makes the roots of the equation $x^2 + 5x + k = 0$ imaginary is

(a) 4 (b) 5 (c) 6 (d) 7

25. The equation of the smallest degree with real coefficients having $1+i$ as one of the roots is

(a) $x^2 + x + 1 = 0$ (b) $x^2 - 2x + 2 = 0$

(c) $x^2 + 2x + 2 = 0$ (d) $x^2 + 2x - 2 = 0$

ANSWERS

1. (b) 2. (b) 3. (c) 4. (d) 5. (d) 6. (b) 7. (c) 8. (a)
9. (b) 10. (a) 11. (a) 12. (b) 13. (c) 14. (c) 15. (b) 16. (a)
17. (c) 18. (a) 19. (b) 20. (b) 21. (d) 22. (b), (c) 23. (c)
24. (d) 25. (b)

SUMMARY

1. Fundamental Theorem of Algebra: Every polynomial equation $f(x) = 0$ has at least one root, real or imaginary (complex).
2. Every polynomial equation $f(x) = 0$ of degree n has exactly n roots real or imaginary.
3. A quadratic equation cannot have more than two roots.
4. If $ax^2 + bx + c = 0, a \neq 0$ is a quadratic equation with real coefficients, then its roots α and β given by

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and, } \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \text{ or, } \alpha = \frac{-b + \sqrt{D}}{2a} \text{ and, } \beta = \frac{-b - \sqrt{D}}{2a}$$

where $D = b^2 - 4ac$ is as the discriminant of the equation.

(i) If $D = 0$, then $\alpha = \beta = -\frac{b}{2a}$

So, the equation has real and equal roots each equal to $-\frac{b}{2a}$.

(ii) If $a, b, c \in Q$ and D is positive and a perfect square, then roots are rational and unequal.

(iii) If $a, b, c \in R$ and D is positive and a perfect square, then the roots are real and distinct.

(iv) If $D > 0$ but it is not a perfect square, then roots are irrational and unequal.

(v) If $D < 0$, then the roots are imaginary and are given by

$$\alpha = \frac{-b + i\sqrt{4ac - b^2}}{2a} \text{ and } \beta = \frac{-b - i\sqrt{4ac - b^2}}{2a}$$

(vi) If $a = 1, b, c \in I$ and the roots are rational numbers, then these roots must be integers.

(vii) If a quadratic equation in x has more than two roots, then it is an identity in x that is $a = b = c = 0$.

(viii) Complex roots of an equation with real coefficients always occur in pairs. However, this may not be true in case of equations with complex coefficients. For example, $x^2 - 2ix - 1 = 0$ has both roots equal to i.

(ix) Surd root of an equation with rational coefficients always occur in pairs like $2 + \sqrt{3}$ and $2 - \sqrt{3}$. However, Surd roots of an equation with irrational coefficients may not occur in pairs. For example, $x^2 - 2\sqrt{3}\,x + 3 = 0$ has both roots equal to $\sqrt{3}$.